

On the Power of Robust Solutions in Two-Stage Stochastic and Adaptive Optimization Problems

Dimitris Bertsimas

Sloan School of Management and Operations Research Center, Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139, dbertsim@mit.edu

Vineet Goyal

Operations Research Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139,
goyalv@mit.edu

We consider a two-stage mixed integer stochastic optimization problem and show that a static robust solution is a good approximation to the fully adaptable two-stage solution for the stochastic problem under fairly general assumptions on the uncertainty set and the probability distribution. In particular, we show that if the right-hand side of the constraints is uncertain and belongs to a *symmetric* uncertainty set (such as hypercube, ellipsoid or norm ball) and the probability measure is also *symmetric*, then the cost of the optimal fixed solution to the corresponding robust problem is at most twice the optimal expected cost of the two-stage stochastic problem. Furthermore, we show that the bound is tight for symmetric uncertainty sets and can be arbitrarily large if the uncertainty set is not symmetric. We refer to the ratio of the optimal cost of the robust problem and the optimal cost of the two-stage stochastic problem as the *stochasticity gap*. We also extend the bound on the stochasticity gap for another class of uncertainty sets referred to as *positive*.

If both the objective coefficients and right-hand side are uncertain, we show that the stochasticity gap can be arbitrarily large even if the uncertainty set and the probability measure are both symmetric. However, we prove that the adaptability gap (ratio of optimal cost of the robust problem and the optimal cost of a two-stage fully adaptable problem) is at most four even if both the objective coefficients and the right-hand side of the constraints are uncertain and belong to a symmetric uncertainty set. The bound holds for the class of *positive* uncertainty sets as well. Moreover, if the uncertainty set is a hypercube (special case of a symmetric set), the adaptability gap is one under an even more general model of uncertainty where the constraint coefficients are also uncertain.

Key words: robust optimization; stochastic optimization; adaptive optimization; stochasticity gap; adaptability gap

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1. Introduction. In most real-world problems, several parameters are uncertain at the optimization phase and a solution obtained through a deterministic optimization approach might be sensitive to even slight perturbations in the problem parameters, possibly rendering it highly suboptimal or infeasible. Stochastic optimization that was introduced as early as Dantzig [10] has been extensively studied in the literature to address uncertainty. A stochastic optimization approach assumes a probability distribution over the uncertain parameters and tries to compute a (two-stage or a multistage) solution that optimizes the expected value of the objective function. We refer the reader to several textbooks including Infanger [15], Kall and Wallace [16], Prékopa [17], Birge and Louveaux [9], Shapiro [20], Shapiro et al. [22] and the references therein for a comprehensive view of stochastic optimization. Whereas a stochastic optimization approach addresses the issue of uncertain parameters, it is by and large computationally intractable. Shapiro and Nemirovski [21] give hardness results for two-stage and multistage stochastic optimization problems where they show that multistage stochastic optimization is computationally intractable even if approximate solutions are desired. Furthermore, to solve a two-stage stochastic optimization problem, Shapiro and Nemirovski [21] present an approximate sampling based algorithm where a sufficiently large number of scenarios (depending on the variance of the objective function and the desired accuracy level) are sampled from the assumed distribution and the solution to the resulting sampled problem is argued to provide an approximate solution to the original problem.

More recently, a robust optimization approach has been introduced to address the problem of optimization under uncertainty and has been studied extensively (see Ben-Tal and Nemirovski [3], Bertsimas and Sim [5, 6]). In a robust optimization approach, the uncertain parameters are assumed to belong to some uncertainty set and the goal is to construct a solution such that the objective value in the worst-case realization of the parameters in the uncertainty set is minimized. A robust optimization approach constructs a single solution that is feasible for all possible realizations of the parameters in the assumed uncertainty set. Therefore, it is a significantly more tractable approach computationally as compared to a stochastic optimization approach. However, it is possible that because a robust optimization approach tries to optimize over the worst-case scenario, it may produce

conservative solutions. We point the reader to the survey by Bertsimas et al. [7] and the references therein for an extensive review of the literature in robust optimization.

To address this drawback of robust optimization, an adaptive optimization approach has been considered where the recourse solution can be adjusted according to the realization of the uncertain parameters where the objective is to minimize the worst-case cost. However, the adaptive problem is intractable in general and approximate adaptive optimization approaches have been considered in the literature where simpler functional forms (such as an affine policy or linear decision rules) are considered to approximate the optimal decisions. The functional form allows to succinctly represent the solution in each stage for every realization of the uncertain parameters, albeit the loss in optimality. This approach was first considered in Rockafellar and Wets [19] in the context of stochastic optimization, and then in robust optimization (Ben-Tal et al. [2]), and extended to linear systems theory (Ben-Tal et al. [1]). In a recent paper, Bertsimas et al. [8] consider a one-dimensional, box-constrained multistage robust optimization problem and show that an affine policy is optimal in this setting. However, in general an affine policy does not necessarily provide a good approximation to the adaptive problem (Bertsimas and Goyal [4]). Moreover, the computation complexity of solving an adaptive optimization problem is significantly higher.

In this paper, we show that under a fairly general model of uncertainty for a two-stage mixed integer optimization problem, a robust optimization approach is a good approximation to solving the corresponding stochastic optimization problem optimally. In other words, the worst-case cost of an optimal solution to the robust two-stage mixed integer optimization problem is not much worse than the expected cost of an optimal solution to the corresponding two-stage stochastic optimization problem when the right-hand side of the constraints is uncertain and belongs to a *symmetric* uncertainty set and the probability distribution is also *symmetric* (we also extend our result under milder conditions). Furthermore, a robust optimization problem can be solved efficiently (as compared to stochastic and adaptive) and thus, provides a computationally tractable approach to obtain good approximations to the two-stage stochastic problem. We also show that a robust optimization approach is an arbitrarily bad approximation to the two-stage stochastic optimization problem when both costs and right-hand sides are uncertain. However, we show that an optimal solution to the robust problem is a good approximation for a two-stage adaptive optimization problem where the goal is to construct a fully-adaptable solution that minimizes the worst-case cost, even when both costs and right-hand sides are uncertain under fairly general assumptions on the uncertainty set.

1.1. Models. We consider the following two-stage stochastic mixed integer optimization problem $\Pi_{\text{Stoch}}(b)$:

$$\begin{aligned}
 z_{\text{Stoch}}(b) = \min_{x, y(\omega) \forall \omega \in \Omega} \quad & c^T x + \mathbb{E}_{\mu}[d^T y(\omega)] \\
 & Ax + By(\omega) \geq b(\omega), \quad \forall \omega \in \Omega, \\
 & x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\
 & y(\omega) \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2},
 \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times n_1}$, $B \in \mathbb{R}^{m \times n_2}$, $c \in \mathbb{R}_+^{n_1}$, $d \in \mathbb{R}_+^{n_2}$, Ω denotes the set of scenarios and for any $\omega \in \Omega$, $b(\omega) \in \mathbb{R}_+^m$ denotes the realization of the uncertain values of right-hand side of the constraints b , and $y(\omega)$ denotes the second-stage decision in scenario ω . Let

$$\mathcal{F}_b(\Omega) = \{b(\omega) \mid \omega \in \Omega\} \subset \mathbb{R}_+^m$$

be the set of possible values of the uncertain parameters (or the uncertainty set), and μ is a probability measure over the set of scenarios Ω . Also, $\mathbb{E}_{\mu}[\cdot]$ is the expectation with respect to the probability measure μ .

The corresponding two-stage robust optimization problem, $\Pi_{\text{Rob}}(b)$ is as follows:

$$\begin{aligned}
 z_{\text{Rob}}(b) = \min_{x, y} \quad & c^T x + d^T y \\
 & Ax + By \geq b(\omega), \quad \forall \omega \in \Omega, \\
 & x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\
 & y \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}.
 \end{aligned} \tag{2}$$

Also, the two-stage adaptive optimization problem, $\Pi_{\text{Adapt}}(b)$ is formulated as follows:

$$\begin{aligned} z_{\text{Adapt}}(b) = \min_{x, y(\omega), \forall \omega \in \Omega} c^T x + \max_{\omega \in \Omega} (d^T y(\omega)) \\ Ax + By(\omega) \geq b(\omega), \quad \forall \omega \in \Omega, \\ x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\ y(\omega) \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}. \end{aligned} \quad (3)$$

We would like to note that the problem $\Pi_{\text{Adapt}}(b)$ is intractable even when there are no integer decision variables. In fact, Feige et al. [12] show that it is hard to approximate $\Pi_{\text{Adapt}}(b)$ within a factor better than $O(\log m)$ even when there are no integer decision variables, i.e., $p_1 = p_2 = 0$, unless $\text{NP} \subset \text{TIME}(2^{O(\sqrt{n})})$, where n is the input size of the problem. Note that we parameterize the problem names with the parameter b that denotes that the right-hand side of the constraints are uncertain. We also extend our uncertainty to include cost uncertainty parametrized as (b, d) . The two-stage stochastic optimization problem, $\Pi_{\text{Stoch}}(b, d)$ under the new model of uncertainty is as follows:

$$\begin{aligned} z_{\text{Stoch}}(b, d) = \min_{x, y(\omega), \forall \omega \in \Omega} c^T x + \mathbb{E}_\mu [d(\omega)^T y(\omega)] \\ Ax + By(\omega) \geq b(\omega), \quad \forall \omega \in \Omega, \\ x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\ y(\omega) \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}, \end{aligned} \quad (4)$$

where $A \in \mathbb{R}^{m \times n_1}$, $B \in \mathbb{R}^{m \times n_2}$, $c \in \mathbb{R}_+^{n_1}$, Ω denotes the set of scenarios, where

$$\mathcal{F}_{(b, d)}(\Omega) = \{(b(\omega), d(\omega)) \mid \omega \in \Omega\} \subset \mathbb{R}_+^{m+n_2}$$

is the uncertainty set and for any $\omega \in \Omega$, $b(\omega) \in \mathbb{R}_+^m$ and $d(\omega) \in \mathbb{R}_+^{n_2}$ are realizations of the uncertain values of right-hand side b , and the second-stage cost vector d , in scenario ω , and μ is a probability measure over the set of scenarios Ω .

The corresponding two-stage robust optimization problem, $\Pi_{\text{Rob}}(b, d)$ is as follows:

$$\begin{aligned} z_{\text{Rob}}(b, d) = \min_{x, y} c^T x + \max_{\omega \in \Omega} (d(\omega)^T y) \\ Ax + By \geq b(\omega), \quad \forall \omega \in \Omega, \\ x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\ y \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}, \end{aligned} \quad (5)$$

and the two-stage adaptive optimization problem, $\Pi_{\text{Adapt}}(b, d)$ is formulated as follows:

$$\begin{aligned} z_{\text{Adapt}}(b, d) = \min_{x, y(\omega), \forall \omega \in \Omega} c^T x + \max_{\omega \in \Omega} (d(\omega)^T y(\omega)) \\ Ax + By(\omega) \geq b(\omega), \quad \forall \omega \in \Omega, \\ x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\ y(\omega) \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}. \end{aligned} \quad (6)$$

The models for the stochastic and the adaptive problems described above, include set covering formulations considered in the recent work on approximation algorithms for two-stage stochastic and robust combinatorial problems such as set cover, facility location and Steiner trees (Immorlica et al. [14], Ravi and Sinha [18], Shmoys and Swamy [23], Gupta et al. [13], Dhamdhere et al. [11]). For instance, by setting A and B to be the element set incidence matrix, we obtain a two-stage set covering problem. We refer the reader to a survey by Swamy and Shmoys [24] on the recent results in approximation algorithms for stochastic combinatorial problems. Our model is more general than the set covering model, as there are no restrictions on the coefficients of the constraint matrices. For instance, we can also model constraints of the form $y(\omega) \leq \theta \cdot x$ that arise in network design problems. This can be done by setting $A = \theta \cdot I$, $B = -I$, and $b = 0$, where I refers to an identity matrix of an appropriate dimension. Note that our model does not admit a complete recourse, unlike the set covering

models considered in the above-mentioned references. Therefore, the sampling-based approaches proposed in Shmoys and Swamy [23] and Gupta et al. [13] do not work for our problem. Although the above models are quite general, we would like to note that the assumed nonnegativity of the right-hand side of the constraints prevents us from modeling packing constraints such as $\sum_{j=1}^n x_j \leq 1$. Moreover, the assumed nonnegativity of the objective coefficients does not allow us to model maximization problems.

Let us also introduce the following definitions before formally describing our contributions.

DEFINITION 1.1. A set $H \subset \mathbb{R}^n$ is a *hypercube*, if there exist $l_i \leq u_i$ for all $i = 1, \dots, n$, such that,

$$H = \{x \in \mathbb{R}^n \mid l_i \leq x_i \leq u_i, \forall i = 1, \dots, n\}.$$

DEFINITION 1.2. A set $P \subset \mathbb{R}^n$ is *symmetric*, if there exists some $u_0 \in P$, such that, for any $z \in \mathbb{R}^n$,

$$(u_0 + z) \in P \Leftrightarrow (u_0 - z) \in P. \tag{7}$$

Note that (7) is equivalent to $x \in P \Leftrightarrow (2u_0 - x) \in P$. A hypercube is a special case of a symmetric set. An ellipsoid, $\mathcal{E}(u, D)$, where $u \in \mathbb{R}^n$ and $D \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix, and

$$\mathcal{E}(u, D) = \{u + D^{1/2}v \mid u, v \in \mathbb{R}^n, v^T v \leq 1\}$$

is also an example of a symmetric set that is a commonly used uncertainty set. Another commonly used uncertainty set that is symmetric is a norm ball $B(x_0, r)$ where $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, and

$$B(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\},$$

where $\|\cdot\|$ denotes some norm (for instance, the euclidean norm). Because most commonly used uncertainty sets are symmetric, our assumption of symmetry on the uncertainty set is not very restrictive. Nevertheless, there are natural uncertainty sets that do not satisfy the assumption of symmetry, such as the following fractional knapsack polytope:

$$P = \left\{ x \in [0, 1]^n \mid \sum_{j=1}^n x_j \leq k \right\}. \tag{8}$$

We show that P is not symmetric even for $n = 2$ and $k = 1$ (see Lemma 2.4). However, P is a natural uncertainty set that occurs in many settings (for instance, in modeling k -fault tolerance). Therefore, it would be useful to prove a bound for such uncertainty sets as well. This motivates us to define the following class of convex sets.

DEFINITION 1.3. A convex set $P \subset \mathbb{R}_+^n$ is *positive* if there exists a convex symmetric set $S \subset \mathbb{R}_+^n$ such that $P \subset S$ and the point of symmetry of S is contained in P .

For example, the convex set P in (8) is positive for $k \geq n/2$. It is contained in a unit hypercube (a symmetric set in the nonnegative orthant) and the point of symmetry of the unit hypercube $\frac{1}{2} \cdot e$ (where e is the vector of all ones) belongs to P when $k \geq n/2$ (see Figure 1). More generally, any polytope in the unit hypercube that contains $\frac{1}{2} \cdot e$ is positive. Another example is the fractional node cover polytope Q of an undirected graph $G = (V, E)$, where $|V| = n$, and

$$Q = \{x \in [0, 1]^n \mid x_i + x_j \geq 1, \forall (i, j) \in E\}.$$

Furthermore, any convex set in the nonnegative orthant that is sufficiently away from the origin is positive. In fact, we can show that all two-dimensional convex sets in the nonnegative orthant are positive. Therefore, our bounds apply to all two-dimensional convex uncertainty sets. Let us also define a symmetric probability measure on a symmetric set.

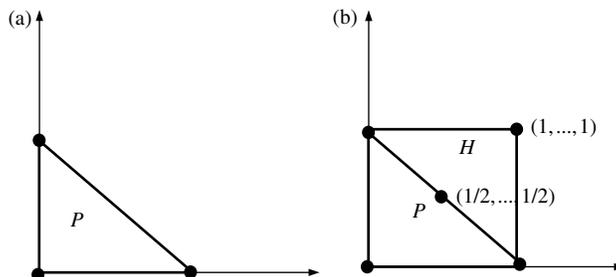


FIGURE 1. (a) $P = \{x \in \mathbb{R}_+^n \mid x_1 + \dots + x_n \leq n/2\}$ is not symmetric. (b) However, P is positive as the unit hypercube $H = [0, 1]^n$ contains P and its point of symmetry, $(1/2, \dots, 1/2)$ belongs to P .

DEFINITION 1.4. A probability measure μ on a symmetric set $P \subset \mathbb{R}^n$, where u^0 is the point of symmetry of P , is *symmetric*, if for any $S \subset P$, $\mu(S) = \mu(\hat{S})$, where $\hat{S} = \{(2u^0 - x) \mid x \in S\}$.

As an example, the uniform probability measure over any symmetric set $P \subset \mathbb{R}^n$ is symmetric.

1.2. Our contributions. In this paper, we compare the optimal cost of the robust problem $\Pi_{\text{Rob}}(b)$ to the optimal costs of problems $\Pi_{\text{Stoch}}(b)$ and $\Pi_{\text{Adapt}}(b)$. We refer to the ratio of $z_{\text{Rob}}(b)$ and $z_{\text{Stoch}}(b)$ (as well as the ratio of $z_{\text{Rob}}(b, d)$ and $z_{\text{Stoch}}(b, d)$) as the *stochasticity gap*, and the ratio of $z_{\text{Rob}}(b)$ and $z_{\text{Adapt}}(b)$ (as well as the ratio of $z_{\text{Rob}}(b, d)$ and $z_{\text{Adapt}}(b, d)$) as the *adaptability gap*. Recall that $z_{\text{Rob}}(b)$, $z_{\text{Stoch}}(b)$, and $z_{\text{Adapt}}(b)$ are the optimal costs of $\Pi_{\text{Rob}}(b)$, $\Pi_{\text{Stoch}}(b)$, and $\Pi_{\text{Adapt}}(b)$, respectively, and $z_{\text{Rob}}(b, d)$, $z_{\text{Stoch}}(b, d)$, and $z_{\text{Adapt}}(b, d)$ are the optimal costs of $\Pi_{\text{Rob}}(b, d)$, $\Pi_{\text{Stoch}}(b, d)$, and $\Pi_{\text{Adapt}}(b, d)$, respectively.

1.2.1. Stochasticity gap. We show that the stochasticity gap is at most two if the uncertainty set is symmetric (see Definition 1.2) as well as the probability distribution over the uncertainty set is *symmetric* (we further extend to other milder conditions on the probability distribution) and there are no integer decision variables in the second stage, i.e., $p_2 = 0$ in $\Pi_{\text{Stoch}}(b)$. This implies that the cost of an optimal fixed solution $x^* \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}$, $y^* \in \mathbb{R}_+^{n_2}$ for $\Pi_{\text{Rob}}(b)$ is at most twice the expected cost of an optimal two-stage solution to $\Pi_{\text{Stoch}}(b)$, and thus the solution x^* , $y(\omega) = y^*$ for all scenarios $\omega \in \Omega$ is a good approximate solution for the problem $\Pi_{\text{Stoch}}(b)$. Moreover, if we use the solution x^* as the first stage solution for $\Pi_{\text{Stoch}}(b)$ and an optimal second-stage solution $y(b)$, given the first stage solution is x^* , then we can show that the expected cost is at most the cost of the static solution (and in many cases strictly better). For all $\omega \in \Omega$, let

$$f(x^*, \omega) = \min\{d^T y \mid By \geq b(\omega) - Ax^*, y \geq 0\}.$$

Because y^* is a feasible second stage solution for all $\omega \in \Omega$ given that the first stage solution is x^* , $f(x^*, \omega) \leq d^T y^*$. Let $f_{\text{Stoch}}(x^*)$ be the optimal expected cost when the first stage solution is x^* . Therefore,

$$f_{\text{Stoch}}(x^*) = c^T x^* + \mathbb{E}_\mu[f(x^*, \omega)] \leq c^T x^* + d^T y^* = z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Stoch}}(b),$$

where the second inequality follows as $f(x^*, \omega) \leq d^T y^*$ for all $\omega \in \Omega$.

Furthermore, an optimal solution to $\Pi_{\text{Rob}}(b)$ can be computed by solving a single mixed integer optimization problem and does not even require any knowledge of the probability distribution μ , although the solution is a good approximation to the stochastic problem only if μ is symmetric. This provides a good computationally tractable approximation to the stochastic optimization problem that is intractable in general. Our results hold under the assumptions of symmetry and nonnegativity on the uncertainty set. Note that most commonly used uncertainty sets, such as hypercubes (specifying an interval of values for each uncertain parameter), ellipsoids and norm balls satisfy these assumptions. The bound on the stochasticity gap holds if the uncertainty set is convex and positive and the probability distribution satisfies a technical condition similar to symmetry. Therefore, we show a surprising approximate equivalence between two-stage robust optimization and two-stage stochastic optimization. The bound on the stochasticity gap is tight for symmetric uncertainty sets and it can be arbitrarily large if the uncertainty set is not symmetric. Therefore, our results give a nice characterization of when a robust solution is a bounded approximation to the stochastic optimization problem with only the right-hand side uncertainty and no integer second-stage decision variables. However, for the model with both cost and right-hand side uncertainty (problems $\Pi_{\text{Stoch}}(b, d)$ and $\Pi_{\text{Rob}}(b, d)$), we show that the stochasticity gap (i.e., the ratio of $z_{\text{Rob}}(b, d)$ and $z_{\text{Stoch}}(b, d)$) can be arbitrarily large even when there are no second-stage integer decision variables and the uncertainty set as well as the probability distribution are symmetric. In fact, the stochasticity gap is large when only the objective coefficients are uncertain and the right-hand side is deterministic.

1.2.2. Adaptability gap. We show that the adaptability gap ($z_{\text{Rob}}(b)/z_{\text{Adapt}}(b)$) is at most two if the uncertainty $\mathcal{F}_b(\Omega)$ is symmetric. The bound of two on the adaptability gap holds even if some of the second-stage decision variables are integers in problems $\Pi_{\text{Adapt}}(b)$ and correspondingly $\Pi_{\text{Rob}}(b)$ unlike the bound on the stochasticity gap which holds only if all the second-stage decision variables are continuous. In fact, the adaptability gap is bounded for problems $\Pi_{\text{Rob}}(b, d)$ and $\Pi_{\text{Adapt}}(b, d)$, where the formulation also models cost uncertainty along with the right-hand side uncertainty unlike the stochasticity gap. Our main results on the adaptability gap are the following:

(i) If the uncertainty set $\mathcal{F}_{(b, d)}(\Omega)$ is a hypercube (which is a special case of a symmetric uncertainty set), then the adaptability gap is one, i.e., $z_{\text{Rob}}(b, d) = z_{\text{Adapt}}(b, d)$. This implies, that there is a single solution (x, y) , $x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}$, $y \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}$ that is feasible for all scenarios $\omega \in \Omega$ and the worst-case cost of

TABLE 1. Stochasticity gap and adaptability gap for different uncertainty sets for the model with uncertain right-hand sides.

Uncertainty set $\mathcal{F}_b(\Omega)$	Stochasticity gap $(z_{\text{Rob}}(b)/(z_{\text{Stoch}}(b)), p_2 = 0)$	Adaptability gap $(z_{\text{Rob}}(b)/(z_{\text{Adapt}}(b)))$
Hypercube	2*	1*
Symmetric	2*	2*
Convex, positive	2	2
Convex	$\Omega(m)$	$\Omega(m)$

Notes. We assume $\mathcal{F}_b(\Omega)$ lies in the nonnegative orthant, objective coefficients $c, d \geq 0$ and $x, y(\omega) \geq 0$. Asterisk denotes that the bound is tight.

this solution is exactly equal to the optimal fully-adaptable solution. In fact, we prove this result for an even more general model of uncertainty where we also allow the constraint coefficients to be uncertain. We would like to note that unlike the adaptability gap, the stochasticity gap is two even if the uncertainty set $\mathcal{F}_b(\Omega)$ is a hypercube and the bound is tight in this case as well.

(ii) For any symmetric uncertainty set $\mathcal{F}_{(b,d)}(\Omega)$, we show that $z_{\text{Rob}}(b, d) \leq 4 \cdot z_{\text{Adapt}}(b, d)$.

(iii) We also extend the bound on the adaptability gap for positive uncertainty sets, i.e., $z_{\text{Rob}}(b, d) \leq 4 \cdot z_{\text{Adapt}}(b, d)$ if $\mathcal{F}_{(b,d)}(\Omega)$ is positive and convex. The bound on the adaptability gap for the case of positive uncertainty sets formalizes the following intuition: the relative change in the optimal cost of a two-stage problem with linear cost function depends on the relative change in the problem parameters and not on the absolute change. If the uncertainty set is positive, all the uncertain parameters are sufficiently far from zero and thus, the relative change in their values can be bounded, a fact that allows us to bound the adaptability gap.

(iv) For a general convex uncertainty set (neither symmetric nor positive), we show that the adaptability gap can be arbitrarily large. In particular, we construct a convex uncertainty set that is neither symmetric nor positive and $z_{\text{Rob}}(b, d) \geq m \cdot z_{\text{Adapt}}(b, d)$. This shows that our results give an almost tight characterization of the uncertainty sets where the adaptability gap is bounded.

Our results on the stochasticity gap and the adaptability gap for the model where only the right-hand side is uncertain are summarized in Table 1.

Table 2 summarizes our results when both right-hand side and objective coefficients are uncertain.

1.2.3. Outline. In §2, we present the bound on the stochasticity gap under symmetric uncertainty sets when only the right-hand side is uncertain. We present examples that show that the bound is tight for this case and also that the bound can be arbitrarily large for general nonsymmetric uncertainty sets in §§2.2 and 2.3. In §2.4, we prove the bound on the stochasticity gap for positive sets. In §3, we show that the stochasticity gap can be arbitrarily large when the objective coefficients are uncertain even when the uncertainty set and the probability distribution are both symmetric and there are no integer decision variables.

In §4, we present our results on the adaptability gap under positive uncertainty sets in the model where only the right-hand side is uncertain. We also present a tight example that shows that the bound on the adaptability gap is tight for a symmetric uncertainty set when only the right-hand side of the constraints is uncertain and an example that shows that the adaptability gap can be arbitrarily large if the uncertainty set is not symmetric. In §5, we prove the bound on the adaptability gap for the model where both cost and right-hand side are uncertain. The special case of hypercube uncertainty is presented in §5.2, where we show that the adaptability gap is one when the uncertainty set is a hypercube even for a more general model where even the constraint coefficients are allowed to be uncertain.

TABLE 2. Stochasticity and adaptability gap for the model with both right-hand side and costs uncertain.

Uncertainty set $\mathcal{F}_{(b,d)}(\Omega)$	Stochasticity gap $(z_{\text{Rob}}(b, d)/(z_{\text{Stoch}}(b, d)))$	Adaptability gap $(z_{\text{Rob}}(b, d)/(z_{\text{Adapt}}(b, d)))$
Hypercube	$\Omega(n_2)$	1*
Symmetric	$\Omega(n_2)$	4
Convex, positive	$\Omega(n_2)$	4
Convex	$\Omega(\max\{m, n_2\})$	$\Omega(m)$

Note. Asterisk denotes that the bound is tight.

2. Stochasticity gap under right-hand side uncertainty. In this section, we consider the robust and stochastic problems $\Pi_{\text{Rob}}(b)$ (cf. (2)) and $\Pi_{\text{Stoch}}(b)$ (cf. (1)), where the right-hand side of the constraints is uncertain. We show that the worst-case cost of the optimal solution of $\Pi_{\text{Rob}}(b)$ is at most two times the expected cost of an optimal solution of $\Pi_{\text{Stoch}}(b)$ if the uncertainty set is symmetric. We also show that the bound is tight for symmetric uncertainty sets and the stochasticity gap can be arbitrarily large if the uncertainty set is not symmetric. We further extend the bound on the stochasticity gap for the case of positive uncertainty sets.

2.1. Symmetric uncertainty sets. In this section, we prove that under fairly general conditions, the stochasticity gap $z_{\text{Rob}}(b)/z_{\text{Stoch}}(b)$ for the two-stage stochastic problem $\Pi_{\text{Stoch}}(b)$ and robust problem $\Pi_{\text{Rob}}(b)$, is at most two for symmetric uncertainty sets. In particular, we prove the following main theorem.

THEOREM 2.1. *Let $z_{\text{Stoch}}(b)$ be the optimal expected cost of $\Pi_{\text{Stoch}}(b)$, and $z_{\text{Rob}}(b)$ be the optimal worst-case cost of the corresponding problem $\Pi_{\text{Rob}}(b)$, where there are no integer decision variables in the second stage, i.e., $p_2 = 0$ and the uncertainty set, $\mathcal{F}_b(\Omega)$ is symmetric. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$ and the probability measure μ on the set of scenarios Ω satisfies that*

$$\mathbb{E}_\mu[b(\omega)] \geq b(\omega^0). \tag{9}$$

Then

$$z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Stoch}}(b).$$

Recall that $\mathbb{E}_\mu[\cdot]$ denotes the expectation with respect to μ , which is a probability measure on the set of scenarios Ω . Because the uncertainty set $\mathcal{F}_b(\Omega)$ is assumed to be symmetric, there exists a point of symmetry, $b^0 \in \mathcal{F}_b(\Omega)$. The scenario where the realization of the uncertain right-hand side is b^0 is referred to as ω^0 , and $b(\omega^0) = b^0$. We require that the expected value of the uncertain right-hand side vector with respect to the probability measure μ is at least $b(\omega^0)$, i.e.,

$$\mathbb{E}_\mu[b_j(\omega)] \geq b_j(\omega^0), \quad j = 1, \dots, m.$$

For instance, consider the following hypercube uncertainty set:

$$\mathcal{F}_b(\Omega) = \{b \in \mathbb{R}_+^m \mid 0 \leq b_j \leq 1, j = 1, \dots, m\}$$

and each component b_j , $j = 1, \dots, m$ takes value uniformly at random between zero and one independent of other components. The point of symmetry of the uncertainty set is $b_j^0 = 1/2$ for all $j = 1, \dots, m$ and it is easy to verify that $\mathbb{E}_\mu[b(\omega)] = b^0$. In fact, (9) is satisfied for any symmetric probability measure μ (see Definition 1.4) on a symmetric uncertainty set as we show in the following lemma.

LEMMA 2.1. *Let μ be a symmetric probability measure on the symmetric set $S \subset \mathbb{R}^n$, where $u^0 \in \mathbb{R}^n$ is the point of symmetry of S . Let x be a random vector drawn from S with respect to the measure μ . Then*

$$\mathbb{E}_\mu[x] = u^0.$$

PROOF. We can write the expectation as follows:

$$\begin{aligned} \mathbb{E}_\mu[x] &= \int_{x \in S} x \, d\mu \\ &= \int_{(2u^0 - y) \in S} (2u^0 - y) \, d\mu \end{aligned} \tag{10}$$

$$= \int_{y \in S} (2u^0 - y) \, d\mu \tag{11}$$

$$\begin{aligned} &= \int_{y \in S} 2u^0 \, d\mu - \int_{y \in S} y \, d\mu \\ &= 2u^0 - \int_{y \in S} y \, d\mu \\ &= 2u^0 - \mathbb{E}_\mu[x], \end{aligned} \tag{12}$$

where (10) follows from a change of variables, setting $y = 2u^0 - x$. Equation (11) follows from the symmetry of S and μ . From (12), we have that $\mathbb{E}_\mu[x] = u^0$. \square

For a symmetric uncertainty set, the symmetry of the probability measure is natural in most practical settings and thus, (9) which is a weaker condition than the symmetry of μ , is not a restrictive assumption. We also generalize the bound on the stochasticity gap to the case where the probability measure does not satisfy (9) but satisfies a weaker assumption (see Theorem 2.2).

Before proving Theorem 2.1, we show an interesting geometric property for symmetric sets. Consider any symmetric set $S \subset \mathbb{R}^n$. For each $j = 1, \dots, n$, let

$$x_j^h = \max_{x \in S} x_j, \quad (13)$$

$$x_j^\ell = \min_{x \in S} x_j. \quad (14)$$

Consider the following hypercube H :

$$H = \{x \in \mathbb{R}^n \mid x_j^\ell \leq x_j \leq x_j^h, \quad j = 1, \dots, n\}. \quad (15)$$

LEMMA 2.2. For any hypercube $H' \subset \mathbb{R}^n$,

$$S \subset H' \Rightarrow H \subset H'.$$

Therefore, H is the smallest hypercube such that $S \subset H$.

PROOF. Let $H' = \{x \in \mathbb{R}^n \mid p_j \leq x_j \leq q_j\}$ such that $S \subset H'$. Consider $x \in H$. Suppose $x \notin H'$. Therefore, there exists $j \in \{1, 2, \dots, n\}$ such that

$$x_j > q_j \quad \text{or} \quad x_j < p_j.$$

We know that $x_j^\ell \leq x_j \leq x_j^h$. Suppose $x_j > q_j$ (the other case is similar). Therefore, $q_j < x_j^h$. Consider

$$\beta \in \arg \max\{x_j \mid x \in S\}.$$

We know that $\beta_j = x_j^h > q_j$. Thus, $\beta \notin H'$, which is a contradiction. \square

LEMMA 2.3. Let x^0 denote the center of hypercube H defined in (15), i.e., for all $j = 1, \dots, n$,

$$x_j^0 = \frac{x_j^\ell + x_j^h}{2}.$$

Then x^0 is the point of symmetry of S . Furthermore, $x \leq 2 \cdot x^0$ for all $x \in S$.

PROOF. Suppose the point of symmetry of S is $u^0 \in S$. Therefore, $(u^0 - z) \in S \Rightarrow (u^0 + z) \in S$ for any $z \in \mathbb{R}^n$. We prove that $u^0 = x^0$. For all $j = 1, \dots, n$, let

$$\begin{aligned} \alpha^j &\in \arg \min\{x_j \mid x \in S\}, \\ \beta^j &\in \arg \max\{x_j \mid x \in S\}. \end{aligned} \quad (16)$$

Note that $\alpha^j = x_j^\ell$ and $\beta^j = x_j^h$ (cf. (13)–(14), see Figure 2). For any $j \in \{1, \dots, m\}$, $\beta^j = u^0 + z^j$ for some $z^j \in \mathbb{R}^n$. Therefore, $u^0 - z^j \in S$, and

$$x_j^\ell \leq u_j^0 - z_j^j = 2u_j^0 - (u_j^0 + z_j^j) = 2u_j^0 - \beta_j^j = 2u_j^0 - x_j^h \Rightarrow \frac{x_j^\ell + x_j^h}{2} \leq u_j^0.$$

Also, $\alpha^j = u^0 - y^j$ for some $y^j \in \mathbb{R}^m$, which implies $(u^0 + y^j) \in S$, and

$$x_j^h \geq u_j^0 + y_j^j = 2u_j^0 - (u_j^0 - y_j^j) = 2u_j^0 - \alpha_j^j = 2u_j^0 - x_j^\ell \Rightarrow \frac{x_j^\ell + x_j^h}{2} \geq u_j^0.$$

Therefore,

$$\frac{x_j^\ell + x_j^h}{2} = u_j^0 = x_j^0, \quad \forall j = 1, \dots, n,$$

which implies that x^0 is the point of symmetry of S . Consider any $x \in S$. For all $j = 1, \dots, n$,

$$x_j \leq x_j^h \leq (x_j^h + x_j^\ell) \leq 2 \cdot x_j^0,$$

where the second last inequality follows from the fact that $x_j^\ell \geq 0$ as $S \subset \mathbb{R}_+^n$. Therefore, $x \leq 2x^0$ for all $x \in S$. \square

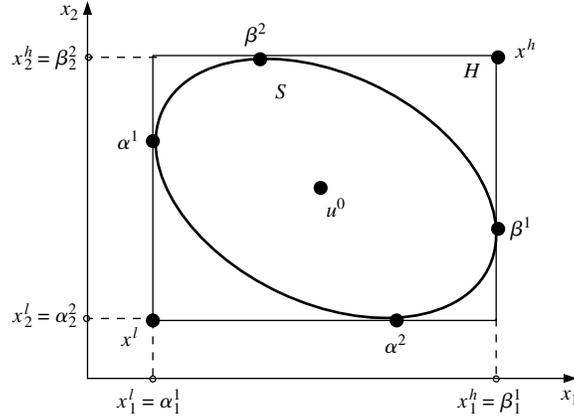


FIGURE 2. A symmetric set S with point of symmetry u^0 , and the bounding hypercube H .
 Note. x^l and x^h are as defined in (13) and (14) and $\alpha^1, \alpha^2, \beta^1, \beta^2$ as defined in (16).

PROOF OF THEOREM 2.1. Consider an optimal solution $x^* \in \mathbb{R}^{n_1-p_1} \times \mathbb{Z}_+^{p_1}$, $y^*(\omega) \in \mathbb{R}_+^{n_2}$ for all $\omega \in \Omega$ for $\Pi_{\text{Stoch}}(b)$. Therefore,

$$A(2x^*) + B(2y^*(\omega^0)) = 2(Ax^* + By^*(\omega^0)) \geq 2b(\omega^0) \tag{17}$$

$$\geq b(\omega), \quad \forall \omega \in \Omega, \tag{18}$$

where (17) follows from the fact that $(x^*, y^*(\omega^0))$ is a feasible solution for scenario ω^0 . Inequality (18) follows as $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$ and $b(\omega) \leq 2b(\omega^0)$ for all $b(\omega) \in \mathcal{F}_b(\Omega)$ from Lemma 2.3. Thus, $(2x^*, 2y^*(\omega^0))$ is a feasible solution for $\Pi_{\text{Rob}}(b)$, and

$$z_{\text{Rob}}(b) \leq c^T(2x^*) + d^T(2y^*(\omega^0)) = 2 \cdot (c^T x^* + d^T y^*(\omega^0)). \tag{19}$$

We know that

$$Ax^* + By^*(\omega) \geq b(\omega), \quad \forall \omega \in \Omega.$$

If we take the expectation of the above inequality with respect to the probability measure μ , we have

$$\mathbb{E}_\mu[Ax^* + By^*(\omega)] \geq \mathbb{E}_\mu[b(\omega)] \geq b(\omega^0).$$

Therefore, by the linearity of expectation,

$$\mathbb{E}_\mu[Ax^*] + \mathbb{E}_\mu[By^*(\omega)] = Ax^* + B\mathbb{E}_\mu[y^*(\omega)] \geq b(\omega^0).$$

Therefore, $\mathbb{E}_\mu[y^*(\omega)]$ is a feasible solution for scenario ω^0 , because there are no integer decision variables in the second stage ($p_2 = 0$). Thus,

$$d^T y^*(\omega^0) \leq d^T \mathbb{E}_\mu[y^*(\omega)], \tag{20}$$

because $y^*(\omega^0)$ is an optimal solution for scenario ω^0 . Also,

$$z_{\text{Stoch}}(b) = c^T x^* + \mathbb{E}_\mu[d^T y^*(\omega)] = c^T x^* + d^T \mathbb{E}_\mu[y^*(\omega)] \tag{21}$$

$$\geq c^T x^* + d^T y^*(\omega^0) \tag{22}$$

$$\geq \frac{z_{\text{Rob}}(b)}{2}, \tag{23}$$

where (21) follows from the linearity of expectation and (22) follows from (20). Inequality (23) follows from (19). \square

Note that although the stochasticity gap is bounded when there are some integer decision variables in the first stage, our bound does not hold in general if there are binary decision variables in the model instead of integer decision variables because we construct a feasible solution to $\Pi_{\text{Rob}}(b)$ by scaling the feasible solution for scenario ω^0 by a factor of two. We require the symmetry of the uncertainty set in proving that the scaled solution $(2x^*, 2y^*(\omega^0))$ corresponding to the scenario ω^0 , is feasible for $\Pi_{\text{Rob}}(b)$ and the condition on the probability measure μ is required to prove that the cost of the fixed solution $(2x^*, 2y^*(\omega^0))$ is not much worse than the optimal expected cost of $\Pi_{\text{Stoch}}(b)$. As noted earlier, the assumptions on the uncertainty set and the probability measure are not very restrictive and hold in many natural settings. Furthermore, the bound on the stochasticity gap generalizes even if the Condition (9) on the probability measure does not hold, although the bound might be worse. In particular, we prove the following theorem that is a generalization of Theorem 2.1.

THEOREM 2.2. *Consider the problems $\Pi_{\text{Stoch}}(b)$ and $\Pi_{\text{Rob}}(b)$ such that there are no integer decision variables in the second stage, i.e., $p_2 = 0$ and the uncertainty set $\mathcal{F}_b(\Omega)$, is symmetric. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$ and suppose for some $\delta > 0$, $\delta \cdot b(\omega^0) \in \mathcal{F}_b(\Omega)$ and the probability measure μ on the set of scenarios Ω satisfies that*

$$\mathbb{E}_\mu[b(\omega)] \geq \delta \cdot b(\omega^0). \quad (24)$$

Then

$$z_{\text{Rob}}(b) \leq \left\lceil \frac{2}{\delta} \right\rceil \cdot z_{\text{Stoch}}(b).$$

PROOF. Let $\tilde{\omega}$ denote the scenario such that $b(\tilde{\omega}) = \delta \cdot b(\omega^0)$. Consider an optimal solution $x^* \in \mathbb{R}^{n_1-p_1} \times \mathbb{Z}_+^{p_1}$, $y^*(\omega) \in \mathbb{R}_+^{n_2}$ for all $\omega \in \Omega$ for $\Pi_{\text{Stoch}}(b)$. We first show that the solution $\lceil 2/\delta \rceil \cdot (x^*, y^*(\tilde{\omega}))$ is a feasible solution for $\Pi_{\text{Rob}}(b)$:

$$\begin{aligned} \left\lceil \frac{2}{\delta} \right\rceil \cdot (Ax^* + By^*(\tilde{\omega})) &\geq \frac{2}{\delta} \cdot (Ax^* + By^*(\tilde{\omega})) \\ &\geq \frac{2}{\delta} \cdot b(\tilde{\omega}) \end{aligned} \quad (25)$$

$$= \frac{2}{\delta} \cdot \delta \cdot b(\omega^0) \quad (26)$$

$$\begin{aligned} &= 2 \cdot b(\omega^0) \\ &\geq b(\omega), \quad \forall \omega \in \Omega, \end{aligned} \quad (27)$$

where (25) follows from the feasibility of $(x^*, y^*(\tilde{\omega}))$ for scenario $\tilde{\omega}$ and (26) follows as $b(\tilde{\omega}) = \delta \cdot b(\omega^0)$ by definition. Inequality (27) follows from the fact that $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$ and from Lemma 2.3, $b(\omega) \leq 2b(\omega^0)$ for all $\omega \in \Omega$. Note that we scale the solution $(x^*, y^*(\omega^0))$ by a factor of $\lceil 2/\delta \rceil$ instead of only $2/\delta$ to preserve the feasibility of integer decision variables in the first stage. Now,

$$z_{\text{Rob}}(b) \leq \left\lceil \frac{2}{\delta} \right\rceil \cdot (c^T x^* + d^T y^*(\tilde{\omega})). \quad (28)$$

Using an argument similar to the proof of Theorem 2.1, we can show that $\mathbb{E}_\mu[y^*(\omega)]$ is a feasible solution for scenario $\tilde{\omega}$ which implies that

$$d^T y^*(\tilde{\omega}) \leq d^T \mathbb{E}_\mu[y^*(\omega)], \quad (29)$$

because $y^*(\tilde{\omega})$ is an optimal solution for scenario $\tilde{\omega}$. Now,

$$\begin{aligned} z_{\text{Stoch}}(b) &= c^T x^* + \mathbb{E}_\mu[d^T y^*(\omega)] \\ &= c^T x^* + d^T \mathbb{E}_\mu[y^*(\omega)] \\ &\geq c^T x^* + d^T y^*(\tilde{\omega}), \end{aligned} \quad (30)$$

where (30) follows from (29) and the bound

$$z_{\text{Rob}}(b) \leq \left\lceil \frac{2}{\delta} \right\rceil \cdot z_{\text{Stoch}}(b)$$

follows from (28). \square

The bound of two on the stochasticity gap can be further improved if $0 \notin \mathcal{F}_b(\Omega)$. Let $b^l, b^h \in \mathbb{R}_+^m$ be such that for all $j = 1, \dots, m$,

$$b_j^l = \min_{b \in \mathcal{F}_b(\Omega)} b_j, \quad b_j^h = \max_{b \in \mathcal{F}_b(\Omega)} b_j. \tag{31}$$

If $b^l \geq \rho \cdot b^h$ for some $0 \leq \rho \leq 1$, we can improve the bound on the stochasticity gap in Theorem 2.2 as

$$b(\omega^0) = \frac{b^l + b^h}{2} \geq \frac{\rho b^h + b^h}{2} = (1 + \rho) \cdot \frac{b^h}{2}.$$

Therefore, if we scale the optimal solution of $\Pi_{\text{Stoch}}(b)$ for scenario ω^0 by a factor $\lceil 2/(1 + \rho) \rceil$ (instead of scaling by a factor 2), we obtain a feasible solution for $\Pi_{\text{Rob}}(b)$, because

$$\begin{aligned} \left\lceil \frac{2}{1 + \rho} \right\rceil \cdot (Ax^* + By^*(\omega^0)) &\geq \frac{2}{1 + \rho} \cdot b(\omega^0) \\ &= \frac{2}{1 + \rho} \cdot \frac{b^l + b^h}{2} \\ &\geq \frac{2}{1 + \rho} \cdot \frac{b^h \cdot (1 + \rho)}{2} \\ &= b^h \\ &\geq b(\omega), \quad \forall \omega \in \Omega. \end{aligned}$$

Combining with the result of Theorem 2.2, we have the following theorem.

THEOREM 2.3. *Consider the problems $\Pi_{\text{Stoch}}(b)$ and $\Pi_{\text{Rob}}(b)$, where there are no integer decision variables in the second stage, i.e., $p_2 = 0$ and the uncertainty set $\mathcal{F}_b(\Omega)$ is symmetric. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$ and suppose for some $\delta > 0$, $\delta \cdot b(\omega^0) \in \mathcal{F}_b(\Omega)$ and the probability measure μ on the set of scenarios Ω satisfies condition (24). Let $b^l, b^h \in \mathbb{R}_+^m$ be as defined in (31) and suppose $b^l \geq \rho \cdot b^h$ for some $\rho \geq 0$. Then*

$$z_{\text{Rob}}(b) \leq \left\lceil \frac{2}{\delta(1 + \rho)} \right\rceil \cdot z_{\text{Stoch}}(b).$$

Note that the condition that $b^l \geq \rho b^h$ is trivially satisfied for $\rho = 0$ because $\mathcal{F}_b(\Omega) \subset \mathbb{R}_+^m$ and $b^l \geq 0$. For $\rho = 0$, we get back the bound of Theorem 2.2. We would like to remark that our results in Theorems 2.1–2.3 hold even for the model without the nonnegativity restrictions on the objective coefficients and the decision variables.

In Theorems 2.1–2.3, we prove that an optimal solution to the robust problem $\Pi_{\text{Rob}}(b)$ is a good approximation for the two-stage stochastic problem $\Pi_{\text{Stoch}}(b)$. We next show that an optimal solution to $\Pi_{\text{Rob}}(b)$ can be computed by solving a single mixed integer optimization problem whose size does not depend on the uncertainty set or the number of worst-case scenarios. In particular, we prove the following theorem.

THEOREM 2.4. *Let $b^h \in \mathbb{R}_+^m$ be such that for each $j = 1, \dots, m$,*

$$b_j^h = \max_{b(\omega) \in \mathcal{F}_b(\Omega)} b_j(\omega).$$

Then the optimal solution to $\Pi_{\text{Rob}}(b)$ can be obtained by solving the following mixed integer problem Π :

$$\begin{aligned} z(\Pi) = \min \quad &c^T x + d^T y, \\ &Ax + By \geq b^h, \\ &x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}, \\ &y \in \mathbb{R}_+^{n_2}. \end{aligned}$$

PROOF. Consider an optimal solution (\hat{x}, \hat{y}) to Π . Clearly, (\hat{x}, \hat{y}) is a feasible solution to $\Pi_{\text{Rob}}(b)$ because

$$A\hat{x} + B\hat{y} \geq b^h \geq b(\omega), \quad \forall \omega \in \Omega.$$

Therefore, $z_{\text{Rob}}(b) \leq z(\Pi)$.

Now, consider an optimal solution (x^*, y^*) to $\Pi_{\text{Rob}}(b)$. We show that it is a feasible solution to Π . For the sake of contradiction, suppose it is not feasible. Therefore, there exists $j \in \{1, \dots, m\}$ such that,

$$(Ax^* + By^*)_j < b_j^h.$$

Let

$$\beta^j \in \arg \max \{b_j(\omega) \mid b(\omega) \in \mathcal{F}_b(\Omega)\}.$$

Therefore,

$$Ax^* + By^* \geq \beta^j \Rightarrow (Ax^* + By^*)_j \geq \beta_j^j = b_j^h,$$

because β^j is a possible realization of the uncertain right-hand side and $\beta_j^j = b_j^h$ (by definition), which is a contradiction. Therefore, $z(\Pi) \leq z_{\text{Rob}}(b)$. \square

Note that the problem Π has only m constraints and $(n_1 + n_2)$ decision variables and thus, the size of Π does not depend on the number of scenarios. Therefore, a good approximate solution to $\Pi_{\text{Stoch}}(b)$ can be computed by solving a single deterministic mixed integer optimization problem whose size does not depend on the uncertainty set and even without the knowledge of the probability distribution, as long as it satisfies (9) or (24), for example.

2.2. A tight stochasticity gap example for symmetric uncertainty sets. Here, we present an instance of $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Stoch}}(b)$, where the uncertainty set $\mathcal{F}_b(\Omega)$ is symmetric such that, $z_{\text{Rob}}(b) = 2z_{\text{Stoch}}(b)$. Consider the following instance where $n_1 = 0, n_2 = n, m = n, A = 0, c = 0, d = (1, \dots, 1), B = I_n$ (here I_n denotes a $n \times n$ identity matrix). Let Ω denote the set of uncertain scenarios and the uncertainty set

$$\mathcal{F}_b(\Omega) = \{b \in \mathbb{R}_+^n \mid 0 \leq b_j \leq 1, j = 1, \dots, n\}.$$

Also, each $b_j, j = 1, \dots, n$ takes a value uniformly at random between zero and one and independent of other components and the probability measure μ is defined according to this distribution. Therefore,

$$\mathbb{E}_\mu[b(\omega)] = (\mathbb{E}_\mu[b_1(\omega)], \dots, \mathbb{E}_\mu[b_n(\omega)]) = (\frac{1}{2}, \dots, \frac{1}{2}).$$

Note that $\mathcal{F}_b(\Omega)$ is a hypercube and thus, a symmetric set in the nonnegative orthant with $b^0 = (1/2, \dots, 1/2)$ as the point of symmetry. Also, $\mathbb{E}_\mu[b(\omega)] = b^0$. Therefore, the uncertainty set $\mathcal{F}_b(\Omega)$, and the probability measure μ , satisfy the assumptions in Theorem 2.1.

THEOREM 2.5. For $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Stoch}}(b)$ defined above,

$$z_{\text{Rob}}(b) = 2 \cdot z_{\text{Stoch}}(b).$$

PROOF. Consider any feasible solution (y_1, \dots, y_n) for $\Pi_{\text{Rob}}(b)$. For any $(b_1, \dots, b_n) \in \mathcal{F}_b(\Omega)$, we require that $y_j \geq b_j$ for all $j = 1, \dots, n$. We know that $(1, \dots, 1) \in \mathcal{F}_b(\Omega)$. Therefore, $y_j \geq 1$ for all $j = 1, \dots, n$, which implies that $z_{\text{Rob}}(b) = n$. Now, consider $\Pi_{\text{Stoch}}(b)$ and consider the solution $\hat{y}(\omega) = b(\omega)$. Clearly, the solution $\hat{y}(\omega)$ for all $\omega \in \Omega$ is feasible. Therefore,

$$\begin{aligned} z_{\text{Stoch}}(b) &\leq \mathbb{E}_\mu[d^T \hat{y}(\omega)] \\ &= \sum_{j=1}^n \mathbb{E}_\mu[\hat{y}_j(\omega)] \\ &= \sum_{j=1}^n \mathbb{E}_\mu[b_j(\omega)] \end{aligned} \tag{32}$$

$$\begin{aligned} &= \sum_{j=1}^n \frac{1}{2} \\ &= \frac{n}{2}, \end{aligned} \tag{33}$$

where (32) follows from the fact that $y(\omega) = b(\omega)$ for all $\omega \in \Omega$. Also, from Theorem 2.1, we have that $z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Stoch}}(b)$. Therefore, $z_{\text{Rob}}(b) = 2 \cdot z_{\text{Stoch}}(b)$. \square

2.3. A large stochasticity gap example for nonsymmetric sets. We show that if the uncertainty set is not symmetric, then the stochasticity gap can be arbitrarily large. Consider the following instance of $\Pi_{\text{Rob}}(b)$, where $n_1 = 0$, $n_2 = m = n \geq 3$, $d = (0, \dots, 0, 1) \in \mathbb{R}_+^n$.

$$\begin{aligned} \min \quad & d^T y, \\ & I_n y \geq b, \quad \forall b \in \mathcal{F}_b(\Omega), \\ & y \geq 0, \end{aligned} \tag{34}$$

where I_n is the $n \times n$ identity matrix. The corresponding instance for $\Pi_{\text{Stoch}}(b)$ is the following:

$$\begin{aligned} \min \quad & \mathbb{E}_\mu[d^T y(b)], \\ & I_n y(b) \geq b, \quad \forall b \in \mathcal{F}_b(\Omega), \\ & y \geq 0, \end{aligned} \tag{35}$$

where Ω denotes the set of scenarios and the uncertainty set

$$\mathcal{F}_b(\Omega) = \left\{ b \in [0, 1]^n \mid \sum_{j=1}^n b_j \leq 1, b \geq 0 \right\}. \tag{36}$$

Also, μ is the uniform probability measure on $\mathcal{F}_b(\Omega)$, i.e., for any $\mathcal{S} \subset \Omega$,

$$\mu(\mathcal{S}) = \frac{\text{volume}(\{b(\omega) \mid \omega \in \mathcal{S}\})}{\text{volume}(\mathcal{F}_b(\Omega))}.$$

THEOREM 2.6. For $\Pi_{\text{Stoch}}(b)$ and $\Pi_{\text{Rob}}(b)$ defined above,

$$z_{\text{Rob}}(b) \geq (n + 1) \cdot z_{\text{Stoch}}(b).$$

We first show that the uncertainty set $\mathcal{F}_b(\Omega)$ is not symmetric.

LEMMA 2.4. The uncertainty set $\mathcal{F}_b(\Omega)$ defined in (36) is not symmetric for $n \geq 2$.

PROOF. For the sake of contradiction, suppose $\mathcal{F}_b(\Omega)$ is symmetric and let u^0 denote the center of $\mathcal{F}_b(\Omega)$. Because $0 = (u^0 - u^0) \in \mathcal{F}_b(\Omega)$, then $(u^0 + u^0) = 2u^0 \in \mathcal{F}_b(\Omega)$. Therefore,

$$\sum_{i=1}^n 2u_i^0 \leq 1 \Rightarrow u_j^0 \leq \frac{1}{2}, \quad j = 1, \dots, n.$$

Let e_j denote the j th unit vector in \mathbb{R}_+^n where only the j th coordinate is one and all others are zero. Now,

$$e_j = u^0 + (e_j - u^0) \in \mathcal{F}_b(\Omega) \Rightarrow x^j = u^0 - (e_j - u^0) \in \mathcal{F}_b(\Omega) \quad \forall j = 1, \dots, n.$$

If there exists $j \in \{1, \dots, n\}$ such that $u_j^0 < \frac{1}{2}$, then

$$x_j^j = u_j^0 - (1 - u_j^0) = 2u_j^0 - 1 < 0,$$

which is a contradiction. Therefore, $u_j^0 = \frac{1}{2}$ for all $j = 1, \dots, n$. Now, $2u^0 = (1, \dots, 1)$, which is a contradiction because $2u^0 \in \mathcal{F}_b(\Omega)$, but $(1, \dots, 1) \notin \mathcal{F}_b(\Omega)$. Therefore, $\mathcal{F}_b(\Omega)$ is not symmetric. \square

PROOF OF THEOREM 2.6. Consider the robust problem $\Pi_{\text{Rob}}(b)$:

$$\begin{aligned} z_{\text{Rob}}(b) = \min_y \quad & d^T y, \\ & I_n y \geq b(\omega), \quad \forall \omega \in \Omega, \\ & y \geq 0. \end{aligned}$$

Because $e_j \in \mathcal{F}_b(\Omega)$ for all $j = 1, \dots, n$, for any feasible solution $y \in \mathbb{R}_+^n$, $I_n y \geq e_j$ for all $j = 1, \dots, n$. Therefore, $y_j \geq 1$ for all $j = 1, \dots, n$, which implies

$$z_{\text{Rob}}(b) \geq 1.$$

Now, consider $\Pi_{\text{Stoch}}(b)$:

$$\begin{aligned} z_{\text{Stoch}}(b) &= \min_y \mathbb{E}_\mu[d^T y(\omega)], \\ I_n y(\omega) &\geq b(\omega), \quad \forall \omega \in \Omega, \\ y(\omega) &\geq 0. \end{aligned}$$

Consider the solution $\hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$. Clearly, the solution $\hat{y}(\omega)$ is feasible, as $I_n \hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$. Now,

$$\begin{aligned} z_{\text{Stoch}}(b) &\leq \mathbb{E}_\mu[d^T \hat{y}(\omega)] \\ &= \mathbb{E}_\mu[\hat{y}_n(\omega)] \\ &= \mathbb{E}_\mu[b_n(\omega)] \end{aligned} \tag{37}$$

$$= \frac{\int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \cdots \int_{x_n=0}^{1-(x_1+x_2+\cdots+x_{n-1})} x_n dx_n dx_{n-1} \cdots dx_1}{\text{volume}(\mathcal{F}_b(\Omega))} \tag{38}$$

$$\begin{aligned} &= \frac{\int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \cdots \int_{x_n=0}^{1-(x_1+x_2+\cdots+x_{n-1})} x_n dx_n dx_{n-1} \cdots dx_1}{\int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \cdots \int_{x_n=0}^{1-(x_1+x_2+\cdots+x_{n-1})} dx_n dx_{n-1} \cdots dx_1} \\ &= \frac{1/(n+1)!}{1/n!} \\ &= \frac{1}{n+1}, \end{aligned} \tag{39}$$

where (37) follows because $\hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$, and the integrals in the numerator and the denominator of (39) follow from standard computation. Therefore, $z_{\text{Rob}}(b) \geq (n+1) \cdot z_{\text{Stoch}}(b)$. \square

The example in Theorem 2.6 shows that if the uncertainty set is not symmetric, then the optimal cost of $\Pi_{\text{Rob}}(b)$ can be arbitrarily large as compared to the optimal expected cost of $\Pi_{\text{Stoch}}(b)$. In fact, this example also shows that the optimal cost of $\Pi_{\text{Adapt}}(b)$ can be arbitrarily large as compared to the optimal cost of $\Pi_{\text{Stoch}}(b)$. Consider the adaptive problem $\Pi_{\text{Adapt}}(b)$ for the instance in Theorem 2.6 and consider the scenario ω' , where $b_n(\omega') = 1$ and $b_j(\omega') = 0, j = 1, \dots, n-1$. Let $y^*(\omega)$ for all $\omega \in \Omega$ be an optimal solution for $\Pi_{\text{Adapt}}(b)$. Therefore, for scenario ω' , $I_n y^*(\omega') \geq b(\omega')$. Thus,

$$y_n^*(\omega') \geq b_n(\omega') = 1.$$

Therefore,

$$z_{\text{Adapt}}(b) = \max_{\omega \in \Omega} d^T y^*(\omega) \geq d^T y^*(\omega') \geq 1,$$

which implies that $z_{\text{Adapt}}(b) \geq (n+1) \cdot z_{\text{Stoch}}(b)$. The large gap between the optimal values of $\Pi_{\text{Stoch}}(b)$ and $\Pi_{\text{Adapt}}(b)$ indicates that allowing a fully adaptable solution is not the only reason for the large gap between $\Pi_{\text{Stoch}}(b)$ and $\Pi_{\text{Rob}}(b)$. Instead, the gap is large because the objective function is an expectation in $\Pi_{\text{Stoch}}(b)$ whereas it is the worst-case in both $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Adapt}}(b)$.

2.4. Stochasticity gap for positive uncertainty sets. In this section, we prove a bound on the stochasticity gap when the uncertainty set is not necessarily symmetric. In view of the large stochasticity gap example for a nonsymmetric uncertainty set, it is clear that we need additional restrictions on the uncertainty set for the stochasticity gap to be bounded. We prove that the stochasticity gap is at most 2 if the uncertainty set is convex and positive but not necessarily symmetric. Recall that a convex set $P \subset \mathbb{R}_+^n$ is *positive* if there is a convex symmetric $S \subset \mathbb{R}_+^n$ such that $P \subset S$ and the point of symmetry of S belongs to P .

THEOREM 2.7. *Consider the robust and stochastic problems $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Stoch}}(b)$. Suppose the uncertainty set, $\mathcal{F}_b(\Omega)$, is convex and positive. Let $\mathcal{B} \subset \mathbb{R}_+^n$ be a symmetric uncertainty set containing $\mathcal{F}_b(\Omega)$ such that the point of symmetry b^0 of \mathcal{B} is contained in $\mathcal{F}_b(\Omega)$. If $\mathbb{E}_\mu[b(\omega)] \geq b^0$, then*

$$z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Stoch}}(b).$$

PROOF. Let ω^0 be the scenario such that $b(\omega^0) = b^0$. Consider an optimal solution $x^*, y^*(\omega), \forall \omega \in \Omega$ to $\Pi_{\text{Stoch}}(b)$. Therefore,

$$Ax^* + By^*(\omega) \geq b(\omega), \quad \forall \omega \in \Omega.$$

Using an argument similar to the proof of Theorem 2.1, we can show that $\mathbb{E}_\mu[y^*(\omega)]$ is a feasible solution for scenario ω^0 , which implies that

$$d^T y^*(\omega^0) \leq d^T \mathbb{E}_\mu[y^*(\omega)], \tag{40}$$

because $y^*(\omega^0)$ is an optimal solution for scenario ω^0 . Also, we can show that the static solution $(2x^*, 2y^*(\omega^0))$ is a feasible solution for $\Pi_{\text{Rob}}(b)$:

$$A(2x^*) + B(2y^*(\omega^0)) \geq 2b(\omega^0) \geq b(\omega), \quad \forall \omega \in \Omega.$$

Hence,

$$\begin{aligned} z_{\text{Rob}}(b) &\leq 2 \cdot (c^T x^* + d^T y^*(\omega^0)) \\ &\leq 2 \cdot (c^T x^* + d^T \mathbb{E}_\mu[y^*(\omega)]) \\ &= 2 \cdot z_{\text{Stoch}}(b), \end{aligned} \tag{41}$$

where (41) follows from (40). \square

3. Stochasticity gap under cost and right-hand side uncertainty. In this section, we show that the stochasticity gap can be arbitrarily large if we consider both cost and right-hand side uncertainty even if the uncertainty set and the probability distribution are both symmetric and there are no integer decision variables. In fact, we construct an example with no right-hand side uncertainty, no integer decision variables, and a single constraint such that the stochasticity gap is arbitrarily large.

THEOREM 3.1. *Consider the following instances of $\Pi_{\text{Stoch}}(b, d)$ and $\Pi_{\text{Rob}}(b, d)$, where $n_1 = 0, n_2 = n, p_2 = 0, m = 1$, and $c = 0, A = 0$ and $B = [1, 1, \dots, 1] \in \mathbb{R}^{1 \times n}$. Let the uncertainty set $\mathcal{F}_{(b,d)}(\Omega) \subset \mathbb{R}_+^{n+1}$ be given by*

$$\mathcal{F}_{(b,d)}(\Omega) = \{(b(\omega), d(\omega)) \mid b(\omega) = 1, 0 \leq d_j(\omega) \leq 1, \forall j = 1, \dots, n, \forall \omega \in \Omega\},$$

and each d_j is distributed uniformly at random between 0 and 1 and independent of other coefficients. Then,

$$z_{\text{Rob}}(b, d) \geq (n + 1) \cdot z_{\text{Stoch}}(b, d).$$

PROOF. Note that the uncertainty set is a hypercube and thus, symmetric. Let ω^0 denote the scenario corresponding to the point of symmetry of the uncertainty set. Therefore, $d(\omega^0) = (1/2, \dots, 1/2)$ and $b(\omega^0) = 1$. Also, $\mathbb{E}_\mu[d(\omega)] = (1/2, \dots, 1/2) = d(\omega^0)$ and thus, the probability distribution also satisfies (9). Consider an optimal solution \hat{y} to $\Pi_{\text{Rob}}(b, d)$. Therefore, $(\hat{y}_1 + \dots + \hat{y}_n) \geq 1$, and

$$z_{\text{Rob}}(b, d) = \max_{\omega \in \Omega} (d(\omega))^T \hat{y} \geq (1, \dots, 1)^T \hat{y} \geq 1,$$

because there is a scenario ω such that $d_j(\omega) = 1$ for all $j = 1, \dots, n$. On the other hand, we show that $z_{\text{Stoch}}(b, d) \leq 1/(n + 1)$. Consider the following solution $\tilde{y}(\omega)$ for all $\omega \in \Omega$ for $\Pi_{\text{Stoch}}(b, d)$, where for all scenarios $\omega \in \Omega$ and $j = 1, \dots, n$,

$$\tilde{y}_j(\omega) = \begin{cases} 1, & \text{if } d_j(\omega) = \min(d_1(\omega), \dots, d_n(\omega)), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to observe that $\tilde{y}(\omega)$ for all $\omega \in \Omega$ is a feasible solution for $\Pi_{\text{Stoch}}(b, d)$. Therefore,

$$\begin{aligned} z_{\text{Stoch}}(b, d) &\leq \mathbb{E}_\mu[(d(\omega))^T \tilde{y}(\omega)] \\ &= \mathbb{E}_\mu[\min(d_1(\omega), \dots, d_n(\omega))] \end{aligned} \tag{42}$$

$$= \frac{1}{n + 1}, \tag{43}$$

where (42) follows from the construction of $\tilde{y}(\omega)$, which implies $(d(\omega))^T \tilde{y}(\omega) = \min(d_1(\omega), \dots, d_n(\omega))$ for all $\omega \in \Omega$. Inequality (43) follows from the computation of expected value of the minimum of n independent random variables each uniformly random between 0 and 1. Therefore, $z_{\text{Rob}}(b, d) \geq (n + 1) \cdot z_{\text{Stoch}}(b, d)$. \square

4. Adaptability gap under right-hand side uncertainty. In this section, we consider the robust and adaptable problems $\Pi_{\text{Rob}}(b)$ (cf. (2)) and $\Pi_{\text{Adapt}}(b)$ (cf. (3)) and show that the worst-case cost of the optimal solution of $\Pi_{\text{Rob}}(b)$ is at most two times the worst-case cost of an optimal adaptable solution of $\Pi_{\text{Adapt}}(b)$ if the uncertainty set is positive. Because positive sets are a generalization of symmetric sets, the result follows for symmetric sets as well. We also show that the bound is tight for even symmetric uncertainty sets and the adaptability gap can be arbitrarily large if the uncertainty set is not symmetric as in the case of the stochasticity gap under right-hand side uncertainty.

4.1. Adaptability gap for positive uncertainty sets. We show that if the uncertainty set $\mathcal{F}_b(\Omega)$ is symmetric, the adaptability gap is at most two. The bound on the adaptability gap holds even when there are integer decision variables in the second stage unlike the case of the stochasticity gap where integer decision variables are allowed only in the first stage. However, the adaptability gap bound does not hold if there are binary decision variables in the model.

Let us first consider the simpler case where there are no integer decision variables in the second stage. The bound on the adaptability gap in this case follows directly from the bound on the stochasticity gap, as for any probability measure μ :

$$z_{\text{Stoch}}(b) \leq z_{\text{Adapt}}(b) \leq z_{\text{Rob}}(b).$$

Now, consider a measure μ that satisfies condition (9) in Theorem 2.1, i.e., $\mathbb{E}_{\mu}[b(\omega)] \geq b(\omega^0)$, where $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$. Clearly, $\mathbb{E}_{\mu}[b(\omega)] = b(\omega^0)$. From Theorem 2.1,

$$z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Stoch}}(b),$$

which implies that

$$z_{\text{Stoch}}(b) \leq z_{\text{Adapt}}(b) \leq z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Stoch}}(b).$$

We prove the bound of 2 on the adaptability gap for the model that allows integer decision variables in the second stage. In particular, we have the following theorem.

THEOREM 4.1. *Let $z_{\text{Rob}}(b)$ denote the optimal worst-case cost of $\Pi_{\text{Rob}}(b)$ and $z_{\text{Adapt}}(b)$ denote the optimal worst-case cost of $\Pi_{\text{Adapt}}(b)$, where the uncertainty set $\mathcal{F}_b(\Omega) \subset \mathbb{R}_+^m$ is positive. Then,*

$$z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Adapt}}(b).$$

PROOF. Because $\mathcal{F}_b(\Omega)$ is positive, there exists a convex symmetric set $S \subset \mathbb{R}_+^m$ such that $\mathcal{F}_b(\Omega) \subset S$ and the point of symmetry of S (say u^0) belongs to $\mathcal{F}_b(\Omega)$. Let ω^0 denote the scenario such that $b(\omega^0) = u^0$. Let $x^*, y^*(\omega), \forall \omega \in \Omega$ be an optimal solution for $\Pi_{\text{Adapt}}(b)$. Then,

$$\begin{aligned} z_{\text{Adapt}}(b) &= c^T x^* + \max_{\omega \in \Omega} d^T y^*(\omega) \\ &\geq c^T x^* + d^T y^*(\omega^0). \end{aligned} \tag{44}$$

Also,

$$\begin{aligned} A(2x^*) + B(2y^*(\omega^0)) &= 2(Ax^* + By^*(\omega^0)) \\ &\geq 2b(\omega^0) \end{aligned} \tag{45}$$

$$\geq b(\omega), \quad \forall \omega \in \Omega, \tag{46}$$

where (45) follows from the feasibility of the solution $(x^*, y^*(\omega^0))$ for scenario ω^0 . Inequality (46) follows from the fact that $b(\omega^0)$ is the point of symmetry of $\mathcal{F}_b(\Omega)$ and from Lemma 2.3, we have that $b(\omega) \leq 2b(\omega^0)$ for all $\omega \in \Omega$. Therefore, $(2x^*, 2y^*(\omega^0))$ is a feasible solution for $\Pi_{\text{Rob}}(b)$, and

$$\begin{aligned} z_{\text{Rob}}(b) &\leq c^T(2x^*) + d^T(2y^*(\omega^0)) \\ &= 2 \cdot (c^T x^* + d^T y^*(\omega^0)) \\ &\leq 2 \cdot z_{\text{Adapt}}(b), \end{aligned} \tag{47}$$

where (47) follows from (44). \square

In fact, we prove a stronger bound on the adaptability gap similar to the bound on the stochasticity gap in Theorem 2.3. We have the following theorem.

THEOREM 4.2. Consider the problems $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Adapt}}(b)$, where the uncertainty set $\mathcal{F}_b(\Omega) \subset \mathbb{R}_+^m$ is positive. Let $S \subset \mathbb{R}_+^m$ be a symmetric set that contains $\mathcal{F}_{(b,d)}(\Omega)$ such that the point of symmetry $u^0 \in \mathcal{F}_{(b,d)}(\Omega)$. Also, let $\omega^0 \in \Omega$ be the scenario such that $(b(\omega^0), d(\omega^0)) = u^0$. Let $b^l, b^h \in \mathbb{R}_+^m$ be such that, for all $j = 1, \dots, m$,

$$b_j^l = \min\{b_j \mid (b, d) \in S\}, \quad b_j^h = \max\{b_j \mid (b, d) \in S\}.$$

Suppose $b^l \geq \rho \cdot b^h$ for some $0 \leq \rho \leq 1$, then

$$z_{\text{Rob}}(b) \leq \left\lceil \frac{2}{1 + \rho} \right\rceil \cdot z_{\text{Adapt}}(b).$$

4.2. A tight adaptability gap example for symmetric uncertainty sets. We show that the bound of 2 on the adaptability gap under symmetric right-hand side uncertainty is also tight, like the bound on the stochasticity gap.

Consider the following instance where $n_1 = 0, n_2 = 2, m = 2, A = 0, c = 0, d = (1, 1)$, and $B = I_2$ (here I_2 denotes a 2×2 identity matrix). Let Ω denote the set of uncertain scenarios and the uncertainty set

$$\mathcal{F}_b(\Omega) = \{b \in \mathbb{R}_+^2 \mid b_1 + b_2 = 1\}.$$

Let us first show that $\mathcal{F}_b(\Omega)$ is symmetric.

LEMMA 4.1. The uncertainty set $\mathcal{F}_b(\Omega)$ is symmetric with center $u^0 = (\frac{1}{2}, \frac{1}{2})$.

PROOF. Consider any $z \in \mathbb{R}^2$ such that $(u^0 + z) \in \mathcal{F}(\Omega)$. Therefore,

$$\left(\frac{1}{2} + z_1\right) + \left(\frac{1}{2} + z_2\right) = 1 \Rightarrow z_1 + z_2 = 0.$$

Also,

$$0 \leq \left(\frac{1}{2} + z_j\right) \leq 1 \Rightarrow -\frac{1}{2} \leq z_j \leq \frac{1}{2}, \quad j = 1, 2.$$

Therefore,

$$0 \leq \frac{1}{2} - z_j \leq 1, \quad j = 1, 2,$$

and

$$\left(\frac{1}{2} - z_1\right) + \left(\frac{1}{2} - z_2\right) = 1 - (z_1 + z_2) = 1 \Rightarrow (u^0 - z) \in \mathcal{F}(\Omega),$$

because $z_1 + z_2 = 0$. \square

THEOREM 4.3. For the robust and adaptable problems $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Adapt}}(b)$ in the above instance,

$$z_{\text{Rob}}(b) = 2 \cdot z_{\text{Adapt}}(b).$$

PROOF. Consider any feasible solution (y_1, y_2) for $\Pi_{\text{Rob}}(b)$. For any $(b_1, b_2) \in \mathbb{R}_+^2$ such that $b_1 + b_2 = 1$, we require that $y_1 \geq b_1$ and $y_2 \geq b_2$. Therefore, $y_1 \geq 1, y_2 \geq 1$ and $z_{\text{Rob}}(b) \geq 2$. Now, consider the adaptable problem $\Pi_{\text{Adapt}}(b, d)$ and consider the solution $\hat{y}(\omega) = b(\omega)$. Clearly, the solution $\hat{y}(\omega)$ for all $\omega \in \Omega$ is feasible. Therefore,

$$\begin{aligned} z_{\text{Adapt}}(b) &\leq \max_{\omega \in \Omega} d^T y(\omega) \\ &= \max_{\omega \in \Omega} y_1(\omega) + y_2(\omega) \\ &= \max_{\omega \in \Omega} b_1(\omega) + b_2(\omega) \\ &\leq 1. \end{aligned}$$

Also, from Theorem 4.1, we have that $z_{\text{Rob}}(b) \leq 2 \cdot z_{\text{Adapt}}(b)$. Therefore, $z_{\text{Rob}}(b) = 2 \cdot z_{\text{Adapt}}(b)$. \square

4.3. A large adaptability gap example for nonsymmetric uncertainty sets. In this section, we construct an example of a nonsymmetric uncertainty set such that the worst-case cost of an optimal robust solution is $\Omega(m)$ times the worst-case cost of an optimal adaptable solution. Therefore, the adaptability gap is $\Omega(m)$ in this case.

THEOREM 4.4. *Consider an instance of problem $\Pi_{\text{Rob}}(b)$ and $\Pi_{\text{Adapt}}(b)$, where, $n_1 = 0$, $n_2 = n \geq 3$, $p_2 = 0$, $A = 0$, $c = 0$, $d = (1, 1, \dots, 1)$ (an n dimensional vector of all ones) and $B = I_n$ (I_n is $n \times n$ identity matrix). Let Ω denote the set of scenarios and the uncertainty set*

$$\mathcal{F}_b(\Omega) = \left\{ b \mid \sum_{j=1}^m b_j \leq 1, b \geq 0 \right\}. \quad (48)$$

Then,

$$z_{\text{Rob}}(b) \geq n \cdot z_{\text{Adapt}}(b).$$

PROOF. From Lemma 2.4, we know that the uncertainty set $\mathcal{F}_b(\Omega)$ is not symmetric. Consider the robust problem $\Pi_{\text{Rob}}(b)$

$$\begin{aligned} z_{\text{Rob}}(b) &= \min_y d^T y, \\ I_n y &\geq b(\omega), \quad \forall \omega \in \Omega, \\ y &\geq 0. \end{aligned}$$

For any feasible solution $y \in \mathbb{R}_+^m$, $I_n y \geq e_j$ for all $j = 1, \dots, m$, where e_j is the unit vector corresponding to the j th column of I_n ($e_j \in \mathcal{F}(\Omega)$). Therefore, $y_j \geq 1$ for all $j = 1, \dots, m$ which implies,

$$z_{\text{Rob}}(b) \geq n.$$

Now, consider $\Pi_{\text{Adapt}}(b)$

$$\begin{aligned} z_{\text{Adapt}}(b) &= \min_{\omega \in \Omega} \max d^T y(\omega), \\ I_n y(\omega) &\geq b(\omega), \quad \forall \omega \in \Omega, \\ y(\omega) &\geq 0. \end{aligned}$$

Consider any scenario $\omega \in \Omega$ and let $b(\omega)$ be the realizations of b in scenario ω . Then,

$$\sum_{j=1}^n b_j(\omega) \leq 1.$$

Consider the following feasible solution $\hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$. Now,

$$\begin{aligned} z_{\text{Adapt}}(b) &\leq \max_{\omega \in \Omega} d^T \hat{y}(\omega) \\ &= \max_{\omega \in \Omega} \sum_{j=1}^n b_j(\omega) \\ &\leq 1. \end{aligned}$$

Therefore, $z_{\text{Rob}}(b) \geq n \cdot z_{\text{Adapt}}(b)$. \square

5. Adaptability gap under right-hand side and cost uncertainty. In this section, we bound the adaptability gap for a more general model of uncertainty where both the right-hand side of the constraints and the objective coefficients are uncertain and the second stage decision variables are allowed to be integers. Unlike the stochasticity gap under cost and right-hand side uncertainty, the adaptability gap is bounded and is at most four when the uncertainty set is positive. Because positive sets are a generalization of symmetric sets, the result follows for symmetric sets as well. Furthermore, we show that the adaptability gap is one for the special case of hypercube uncertainty sets in an even more general model of uncertainty that allows constraint coefficients to be uncertain. This result is particularly surprising because the bound of two on the stochasticity gap is tight for hypercube right-hand side uncertainty (cf. §2.2).

5.1. Positive uncertainty sets. We consider problems $\Pi_{\text{Adapt}}(b, d)$ (cf. (6)) and $\Pi_{\text{Rob}}(b, d)$ (cf. (5)), and show that the adaptability gap is at most four if the uncertainty set is positive. In particular, we prove the following theorem.

THEOREM 5.1. *Consider the problems $\Pi_{\text{Rob}}(b, d)$ and $\Pi_{\text{Adapt}}(b, d)$, where the uncertainty set $\mathcal{F}_{(b,d)}(\Omega) \subset \mathbb{R}_+^{m+n_2}$ is positive. Then,*

$$z_{\text{Rob}}(b, d) \leq 4 \cdot z_{\text{Adapt}}(b, d).$$

PROOF. Because $\mathcal{F}_{(b,d)}(\Omega)$ is positive, there exists a convex symmetric set $S \subset \mathbb{R}_+^{m+n_2}$ such that $\mathcal{F}_{(b,d)}(\Omega) \subset S$ and the point of symmetry of S (say u^0) belongs to $\mathcal{F}_{(b,d)}(\Omega)$. Let ω^0 be the scenario corresponding to $u^0 = (b(\omega^0), d(\omega^0))$. Consider an optimal solution $x^*, y^*(\omega), \forall \omega \in \Omega$ to $\Pi_{\text{Adapt}}(b, d)$. We show that the solution $(2x^*, 2y^*(\omega^0))$ is a feasible solution for the robust problem $\Pi_{\text{Rob}}(b, d)$ because

$$A(2x^*) + B(2y^*(\omega^0)) \geq 2b(\omega^0) \geq b(\omega), \quad \forall \omega \in \Omega, \quad (49)$$

where inequality (49) follows from the fact that $(b(\omega^0), d(\omega^0))$ is the center of the symmetric set S that contains $\mathcal{F}_{(b,d)}(\Omega)$. Therefore, $b(\omega) \leq 2b(\omega^0)$ and $d(\omega) \leq 2d(\omega^0)$ for all $\omega \in \Omega$ from Lemma 2.3. Now,

$$\begin{aligned} z_{\text{Rob}}(b, d) &\leq c^T(2x^*) + \max_{\omega \in \Omega} d(\omega)^T(2y^*(\omega^0)) \\ &\leq 2 \cdot c^T x^* + (2 \cdot d(\omega^0))^T(2y^*(\omega^0)) \end{aligned} \quad (50)$$

$$\begin{aligned} &\leq 4 \cdot (c^T x^* + d(\omega^0)^T y^*(\omega^0)) \\ &\leq 4 \cdot z_{\text{Adapt}}(b, d), \end{aligned} \quad (51)$$

where (50) follows from the fact that $d(\omega) \leq 2d(\omega^0)$ for all $\omega \in \Omega$ because $d(\omega^0)$ is the point of symmetry of $S \supset \mathcal{F}_{(b,d)}(\Omega)$, and finally (51) follows because $c^T x^* + d(\omega^0)^T y^*(\omega^0) \leq z_{\text{Adapt}}(b, d)$. \square

Similar to the bound on the adaptability gap under right-hand side uncertainty in Theorem 4.2, we obtain a stronger multiplicative bound on the adaptability gap under cost and right-hand side uncertainty. In particular, we have the following theorem.

THEOREM 5.2. *Consider the problems $\Pi_{\text{Rob}}(b, d)$ and $\Pi_{\text{Adapt}}(b, d)$ where the uncertainty set $\mathcal{F}_{(b,d)}(\Omega)$ is positive. Let $S \subset \mathbb{R}_+^{m+n_2}$ be a symmetric set that contains $\mathcal{F}_{(b,d)}(\Omega)$ such that the point of symmetry $u^0 \in \mathcal{F}_{(b,d)}(\Omega)$. Also, let $\omega^0 \in \Omega$ be the scenario such that $(b(\omega^0), d(\omega^0)) = u^0$. Let $(b^l, d^l), (b^h, d^h) \in \mathbb{R}_+^{m+n_2}$ be such that for all $j = 1, \dots, m$,*

$$b_j^l = \min\{b_j \mid (b, d) \in S\}, \quad b_j^h = \max\{b_j \mid (b, d) \in S\},$$

and for all $j = 1, \dots, n_2$,

$$d_j^l = \min\{d_j \mid (b, d) \in S\}, \quad d_j^h = \max\{d_j \mid (b, d) \in S\}.$$

Suppose $b^l \geq \rho \cdot b^h$ and $d^l \geq \rho \cdot d^h$ for some $0 \leq \rho \leq 1$. Then

$$z_{\text{Rob}}(b, d) \leq \left(\frac{2}{1+\rho} \right) \cdot \left[\frac{2}{1+\rho} \right] \cdot z_{\text{Adapt}}(b, d).$$

PROOF. From Lemma 2.3, we know that

$$b(\omega^0) = \frac{b^l + b^h}{2}, \quad d(\omega^0) = \frac{d^l + d^h}{2}.$$

Because $b^l \geq \rho \cdot b^h$ and $d^l \geq \rho \cdot d^h$, we have that

$$b^h \leq \frac{2}{1+\rho} \cdot b(\omega^0), \quad d^h \leq \frac{2}{1+\rho} \cdot d(\omega^0). \quad (52)$$

Let $x^*, y^*(\omega), \forall \omega \in \Omega$ be an optimal solution for $\Pi_{\text{Adapt}}(b, d)$. Then, we show that the solution

$$\left[\frac{2}{1+\rho} \right] \cdot (x^*, y^*(\omega^0)),$$

is a feasible solution for $\Pi_{\text{Rob}}(b, d)$.

$$\begin{aligned} \left\lceil \frac{2}{1+\rho} \right\rceil \cdot (Ax^* + By^*(\omega^0)) &\geq \frac{2}{1+\rho} \cdot b(\omega^0) \\ &\geq b^h \\ &\geq b(\omega), \quad \forall \omega \in \Omega, \end{aligned} \tag{53}$$

where (53) follows from (52). Therefore,

$$\begin{aligned} z_{\text{Rob}}(b, d) &\leq \left\lceil \frac{2}{1+\rho} \right\rceil \cdot (c^T x^* + \max_{\omega \in \Omega} d(\omega)^T y^*(\omega^0)) \\ &\leq \left\lceil \frac{2}{1+\rho} \right\rceil \cdot (c^T x^* + (d^h)^T y^*(\omega^0)) \\ &\leq \left\lceil \frac{2}{1+\rho} \right\rceil \cdot \left(c^T x^* + \left(\frac{2}{1+\rho} \right) \cdot d(\omega^0)^T y^*(\omega^0) \right) \end{aligned} \tag{54}$$

$$\begin{aligned} &\leq \left(\frac{2}{1+\rho} \right) \cdot \left\lceil \frac{2}{1+\rho} \right\rceil \cdot (c^T x^* + d(\omega^0)^T y^*(\omega^0)) \\ &\leq \left(\frac{2}{1+\rho} \right) \cdot \left\lceil \frac{2}{1+\rho} \right\rceil \cdot z_{\text{Adapt}}(b, d), \end{aligned} \tag{55}$$

where (54) follows from (52) and (55) follows from the fact that $\omega^0 \in \Omega$ and the optimality of $x^*, y^*(\omega)$ for $\Pi_{\text{Adapt}}(b, d)$. \square

5.2. Special case of hypercube uncertainty sets. In this section, we consider the case where the uncertainty set is a hypercube and show that the worst-case cost of the optimal solution of $\Pi_{\text{Adapt}}(b, d)$ is equal to the worst-case cost of the optimal solution of $\Pi_{\text{Adapt}}(b, d)$. In fact, we prove the result for a more general model of uncertainty, one where the coefficients in the constraint matrix are also uncertain. We consider the following adaptive two-stage mixed integer problem, $\Pi_{\text{Adapt}}(A, B, b, d)$:

$$\begin{aligned} z_{\text{Adapt}}(A, B, b, d) &= \min c^T x + \max_{\omega \in \Omega} d(\omega)^T y(\omega), \\ A(\omega)x + B(\omega)y &\geq b(\omega), \quad \forall \omega \in \Omega, \\ x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1}, \\ y(\omega) &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2}, \quad \forall \omega \in \Omega, \end{aligned} \tag{56}$$

where Ω is the set of uncertain scenarios and $\omega \in \Omega$ refers to a particular scenario. Let $\mathcal{U} = \{(A(\omega), B(\omega), b(\omega), d(\omega)) \mid \omega \in \Omega\}$ denote the hypercube uncertainty set for scenarios in Ω , where $A(\omega) \in \mathbb{R}^{m \times n_1}$, $B(\omega) \in \mathbb{R}^{m \times n_2}$, $b(\omega) \in \mathbb{R}^m$, and $d(\omega) \in \mathbb{R}_+^{n_2}$ are realizations of the uncertain parameters in scenario ω .

The robust counterpart $\Pi_{\text{Rob}}(A, B, b, d)$ is formulated as follows:

$$\begin{aligned} z_{\text{Rob}}(A, B, b, d) &= \min c^T x + \max_{\omega \in \Omega} d(\omega)^T y, \\ A(\omega)x + B(\omega)y &\geq b(\omega), \quad \forall \omega \in \Omega, \\ x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1}, \\ y &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2}. \end{aligned} \tag{57}$$

THEOREM 5.3. *Let \mathcal{U} be a hypercube, i.e.,*

$$\mathcal{U} = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_N, u_N],$$

for some $l_i \leq u_i$ for all $i = 1, \dots, N$, where $N = mn_1 + mn_2 + m + n_2$, then

$$z_{\text{Rob}}(A, B, b, d) = z_{\text{Adapt}}(A, B, b, d).$$

PROOF. Consider an optimal solution (x^*, y^*) of the problem $\Pi_{\text{Rob}}(A, B, b, d)$. Therefore,

$$A(\omega)x^* + B(\omega)y^* \geq b(\omega), \quad \forall \omega \in \Omega.$$

Therefore, the solution $x = x^*, y(\omega) = y^*, \forall \omega$ is a feasible solution for $\Pi_{\text{Adapt}}(A, B, b, d)$ which implies,

$$z_{\text{Adapt}}(A, B, b, d) \leq c^T x^* + \max_{\omega \in \Omega} d(\omega)^T y^*.$$

Now consider an optimal solution \hat{x} and $\hat{y}(\omega), \forall \omega \in \Omega$ for $\Pi_{\text{Adapt}}(A, B, b, d)$. Consider the following realization of the uncertain parameters.

$$\bar{A}_{ij} = \min_{\omega \in \Omega} A_{ij}(\omega), \quad \forall i = 1, \dots, m, j = 1, \dots, n_1, \quad (58)$$

$$\bar{B}_{ij} = \min_{\omega \in \Omega} B_{ij}(\omega), \quad \forall i = 1, \dots, m, j = 1, \dots, n_2, \quad (59)$$

$$\bar{b}_i = \max_{\omega \in \Omega} b_i(\omega), \quad \forall i = 1, \dots, m, \quad (60)$$

$$\bar{d}_j = \max_{\omega \in \Omega} d_j(\omega), \quad \forall j = 1, \dots, n_2.$$

Because the uncertainty set is a hypercube, the $\text{vec}(\bar{A}, \bar{B}, \bar{b}, \bar{d})$ is a valid scenario. Let us refer to this scenario as $\bar{\omega}$. Clearly,

$$z_{\text{Adapt}}(A, B, b, d) = c^T \hat{x} + \max_{\omega \in \Omega} d(\omega)^T \hat{y}(\omega) \geq c^T \hat{x} + d(\bar{\omega})^T \hat{y}(\bar{\omega}).$$

We claim that $(\hat{x}, \hat{y}(\bar{\omega}))$ is a feasible solution for $\Pi_{\text{Rob}}(A, B, b, d)$. Consider any $\omega \in \Omega$. Now,

$$A(\omega)\hat{x} + B(\omega)\hat{y}(\bar{\omega}) \geq \bar{A}\hat{x} + \bar{B}\hat{y}(\bar{\omega}) \quad (61)$$

$$\geq b(\bar{\omega}) \quad (62)$$

$$\geq b(\omega), \quad (63)$$

where (61) follows from (58) and (59) and the fact that $\hat{x}, \hat{y}(\bar{\omega}) \geq 0$. Inequality (62) follows from the feasibility of $(\hat{x}, \hat{y}(\bar{\omega}))$ for scenario $\bar{\omega}$ and (60). Inequality (63) follows from the fact that $b(\bar{\omega}) \geq b(\omega)$ for any $\omega \in \Omega$. Therefore,

$$\begin{aligned} z_{\text{Rob}}(A, B, b, d) &\leq c^T \hat{x} + \max_{\omega \in \Omega} d(\omega)^T \hat{y}(\bar{\omega}) \\ &\leq c^T \hat{x} + d(\bar{\omega})^T \hat{y}(\bar{\omega}) \\ &\leq z_{\text{Adapt}}(A, B, b, d), \end{aligned} \quad (64)$$

where inequality (64) follows from the fact that $d(\bar{\omega}) \geq d(\omega)$ for all $\omega \in \Omega$ and $\hat{y}(\bar{\omega}) \geq 0$. \square

Therefore, we obtain the following corollary.

COROLLARY 5.1. *Let $z_{\text{Rob}}(b, d)$ be an optimal solution for $\Pi_{\text{Rob}}(b, d)$ and $z_{\text{Adapt}}(b, d)$ be an optimal solution for $\Pi_{\text{Adapt}}(b, d)$, where the uncertainty set, $\mathcal{F}_{(b,d)}(\Omega)$, is a hypercube. Then,*

$$z_{\text{Rob}}(b, d) = z_{\text{Adapt}}(b, d).$$

6. Conclusions. In this paper, we study the effectiveness of a static-robust solution in approximating two-stage stochastic and adaptive optimization problems and present several surprising positive results under mild restrictions on the model and the uncertainty set. We show that, under fairly general assumptions for the uncertainty set, and the probability measure (namely that both are symmetric), a static-robust solution is a good approximation for the two-stage stochastic optimization problem when the right-hand side of the constraints is uncertain and a good approximation for the two-stage adaptive optimization problem when both the right-hand side and the costs are uncertain. In other words, both the stochasticity gap (when only the right-hand side is uncertain) and the adaptability gap (when both the right-hand side and the costs are uncertain) are bounded. We also show that our bounds on the stochasticity and the adaptability gaps are tight for symmetric uncertainty sets.

The assumption of symmetry on the uncertainty set is not very restrictive and is satisfied by most commonly used uncertainty sets such as hypercubes, ellipsoids and norm balls. Furthermore, we show that if the assumption of symmetry on the uncertainty set is relaxed, both the stochasticity gap and the adaptability gap can be arbitrarily large. We also extend the bounds for positive uncertainty sets that are not necessarily symmetric. Therefore, the results in this paper show that the robust optimization approach is more powerful than believed previously and provides a tractable and good approximation for both the two-stage stochastic and the two-stage adaptive optimization problem under fairly general assumptions.

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